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OPTIMALITY CONDITIONS IN AXISYMMETRIC PROBLEMS OF ELASTICITY THEORY*

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The result /1/ of the optimality of equi-stressed contours in a loaded plane is extended to the case of a space with cavities that is axisymmetric. Boundary conditions are found to determine the shape of the optimal cavities, the inverse axisymmetric problem of elasticity theory, and its analytic solution is obtained in the case of a single cavity.

Let an elastic space S be weakened by a set of n closed cavities that are symmetric relative to the z axis of a cylindrical coordinate system (r, z, θ). Normal pressure of constant intensity p acts on the cavity surface $\Gamma_k\ (k = 1, 2, \ldots, n)$, and a homogeneous stress field is given at infinity

$$\sigma_r^\infty = \sigma_\theta^\infty = q_1, \quad \sigma_z^\infty = q_2; \quad q_1, q_2 > 0 \tag{1}$$

Let a section through Σ of the domain S by a meridian plane (r, z) be denoted by the section $\Gamma_k - \gamma_k$, $k = 1, 2, \ldots, n$ (Fig.1), and the union of γ_k and Γ_k by γ and Γ, respectively.

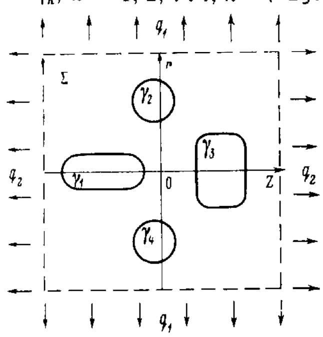

Fig.1

The state of stress at each point of $(S + \Gamma)$ is characterized by the value of the positive function F of the stress tensor invariants I_1, I_2, proportional to the Mises plasticity criterion

$$F(I_1, I_2) = I_1^2 + 3I_2$$

The function F evidently depends also on the space variables r, z.

The stresses in $(S + \Gamma)$ are optimal if F_0, the maximum in the domain of $F(r, z)$, reaches the minimally possible value for a certain boundary /1/:

$$f = \min_\Gamma F_0 = \min_\Gamma \max_{r,z} F(r, z) \tag{2}$$
$$r, z \in \Sigma + \gamma$$

Furthermore, condition (2) is also considered with other functions instead of F. The boundaries at which it is satisfied will be called minimaxes relative to the appropriate functions. Therefore, the inverse problem for the domain under consideration is to seek surfaces that are minimal relative to $F(r, z)$.

To do this we represent the stress field in $(S + \Gamma)$ in the form of the sum of the homogeneous field (1) and the perturbations induced by the cavities. Because of axial symmetry, only the components $\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}$ of the perturbed state are different from zero. They decrease at infinity, and satisfy the following boundary conditions on γ /2/:

$$\sigma_z \frac{dr}{ds} - \tau_{rz} \frac{dz}{ds} = -(q_2 + p)\frac{dr}{ds} \tag{3}$$
$$\tau_{rz}\frac{dr}{ds} - \sigma_r \frac{dz}{ds} = (q_1 + p)\frac{dz}{ds}$$

where s is arclength of the contour γ_k, $k = 1, 2, \ldots, n$.

The invarient I_1 is now written in the form

$$I_1 = 2q_1 + q_2 + \frac{2G(1-\nu)}{1-2\nu}\vartheta$$

(G, ν are elastic constants of the medium, ϑ is the relative volume expansion of the perturbed state that decreases at infinity). In the absence of volume forces, the functions $I_1(r, z)$ and $\vartheta(r, z)$ are harmonic in S /3/.

Relying on the maximum principle for harmonic functions, we obtain the inequality

$$\max |I_1(r, z)| > (I_1)_\infty = 2q_1 + q_2; \quad r, z \in \gamma \tag{4}$$

*Prikl.Matem.Mekhan.,46,No.2,pp.278-282,1982 where the equality sign is achieved only in case $I_1 = \text{const}$ in $(\Sigma + \gamma)$. In the plane case, an estimate of the type (4) is presented in /1,4/.

It follows from (4) that the surfaces Γ_k $(k = 1, 2, \ldots, n)$ are minimaxes with respect to $|I_1(r, z)|$ if the following condition is satisfied in $(\Sigma + \gamma)$

$$\vartheta(r, z) = 0 \tag{5}$$

Let Λ_k denote such surfaces while Γ_k will be kept to denote arbitrary surfaces. Evidently, Λ_k are minimaxes also with respect to the superharmonic function $(I_1(r, z) + c)^2$ which does not achieve a maximum at internal points of $(S + \Gamma)$, and c is an arbitrary constant.

We now find the stresses on Λ. Under the condition (5) the field of perturbations in $(S + \Gamma)$ is described by the equations /2/

$$\frac{\partial(ur)}{\partial z} = r\frac{\partial w}{\partial r}; \quad \frac{\partial(ur)}{\partial r} = -r\frac{\partial w}{\partial z} \tag{6}$$

$$\sigma_z = 2G\frac{\partial w}{\partial z}, \quad \sigma_r = 2G\frac{\partial u}{\partial r}, \quad \sigma_\theta = 2G\frac{u}{r}, \quad \tau_{rz} = G\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \tag{7}$$

(u, w are the displacements in a cylindrical coordinate system). Substituting (7) into (3) and using (6), we have on λ_k the meridian section of Λ_k

$$2G\left[\frac{\partial(ur)}{\partial r}\frac{dr}{ds} + \frac{\partial(ur)}{\partial z}\frac{dz}{ds}\right] = (p + q_2)r\frac{dr}{ds}$$

$$2G\left[\frac{\partial w}{\partial r}\frac{dr}{ds} + \frac{\partial w}{\partial z}\frac{dz}{ds}\right] = (q_1 + p)\frac{dz}{ds} - \frac{u}{r}\frac{dz}{ds}$$

Alternately, integrating these identities with respect to s along λ_k, we obtain

$$u = \frac{q_2 + p}{4G}r + \frac{G_k}{r}$$

$$w = \frac{2q_1 - q_2 + p}{4G}z + D_k + G_k\int_0^s \frac{dz(\xi)}{r(\xi)}; \quad z, r \in \lambda_k$$

(G_k, D_k are constants of integration). It can be shown that G_k equal zero because of the symmetry of the problem and the uniqueness of $w(r, z)$, hence

$$u = \frac{q_2 + p}{4G}r, \quad w = \frac{2q_1 - q_2 + p}{4G}z + D_k \tag{8}$$

It follows from (8) that in a local coordinate system (n, t, θ) on Λ_k

$$\sigma_\theta = 2G\frac{u}{r} = \frac{q_2 + p}{2}$$

(n, t are the arclengths along the positive normal and tangent to λ_k).

Returning to the original state of stress with components $\sigma_\theta^0, \sigma_t^0, \sigma_n^0$, we have on Λ_k $(\sigma_n^0 = -p)$:

$$\sigma_\theta^0 = \sigma_\theta^\infty + \sigma_\theta = \frac{2q_1 + q_2 + p}{2} \tag{9}$$

$$\sigma_t^0 = I_1 - \sigma_n^0 - \sigma_\theta^0 = \frac{2q_1 + q_2 + p}{2}$$

where by symmetry $\tau_{nt}^0, \tau_{n\theta}^0, \tau_{t\theta}^0 = 0$.

Therefore, the stresses σ_θ^0 and σ_t^0 take constant values on the system of surfaces Λ_k defined by the condition (5). From the above it follows that these values are equal. By analogy with the plane problem /5/, we call such surfaces equistressed.

Under the condition (5), the invariant

$$I_2 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_x\sigma_z = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2 + \frac{1}{2}[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - I_1^2]$$

and the functions F are superharmonic functions of the space variables. The proof simplifies in a Cartesian coordinate system.

Taking into account that for $I_1 = \text{const}$ all the stress tensor components are harmonic functions /3/, we obtain (∇^2 is the Laplace operator)

$$\nabla^2 I_2 = 2[(\nabla\tau_{xy})^2 + (\nabla\tau_{yz})^2 + (\nabla\tau_{xz})^2 + (\nabla\sigma_x)^2 + (\nabla\sigma_y)^2 + (\nabla\sigma_z)^2] \geqslant 0$$

from which it follows that

$$\nabla^2 F = \nabla^2 I_1^2 + 3\nabla^2 I_2 = 3\nabla^2 I_2 \geqslant 0$$

Therefore, $F(r, z)$ does not reach the maximum at internal points of the domain /6/.
On the arbitrary boundary Γ, the estimate $(r, z \in \gamma)$

$$\begin{aligned} 2F(r, z) &= (\sigma_n^0 - \sigma_t^0)^2 + (\sigma_n^0 - \sigma_\theta^0)^2 + (\sigma_t^0 - \sigma_\theta^0)^2 \geqslant \\ &(\sigma_n^0 - \sigma_t^0)^2 + (\sigma_n^0 - \sigma_\theta^0)^2 = 2p^2 + 2p(\sigma_t^0 + \sigma_\theta^0) + (\sigma_t^0)^2 + (\sigma_\theta^0)^2 \geqslant \\ &4p^2 + 2pI_1 + \frac{(\sigma_t^0 + \sigma_\theta^0)^2}{2} = \frac{(I_1 + 3p)^2}{2} \end{aligned} \tag{10}$$

is valid for F. In deriving (10) the following known inequality was used (a, b are arbitrary real numbers)

$$2(a^2 + b^2) \geqslant (a + b)^2$$

There results from (9) that the equality signs are reached in (10) on Λ. The proof of the optimality of these surfaces is performed by a chain of inequalities constructed by a scheme proposed in /1/ (t is a point in $\Sigma + \gamma$, τ is on γ, t_0 in $\Sigma + \lambda$ and τ_0 on λ):

$$\max F(t) \geqslant \max F(\tau) \geqslant \max F(\tau_0) \geqslant F(t_0) \geqslant (F)_\infty = (q_1 - q_2)^2$$

where $f = 1/4\,(2q_1 + q_2 + 3p)^2$.

The gain from applying optimal contours can be estimated by means of the quantity $\alpha = \sqrt{F_m/f}$, where F_m is the maximum value of F for the domains being equated.

Thus, by using the solution of the direct problem for a space with a toroidal cavity /2/, we obtain that $\alpha \geqslant 1.42$ for $p = 0$, $q_1 = q_2 = 1$, $r_1/r_2 = 0.5$.

Here r_1 is the radius of the generating circle of the torus, r_2 is the distance between its center and the axis of symmetry. The quantity F_m had the lower estimate in terms of the maximum F on the domain boundary.

Actual seeking of the optimal contours is associated with great difficulties and in the general case reduces to the solution of an inverse boundary value problem for the elliptic system (6), which degenerates onto the axis r with the condition (8) given on the unknown boundary in the (r, z) plane.

For $n = 1$ an ellipsoid of revolution (spheroid) is optimal. Setting $q_1 \geqslant q_2$, $p = 0$ for definiteness, we prove this by using the solution of the direct problem of elasticity theory for the exterior of a compressed spheroid $s = s_0$ under the load (1) in elliptic coordinates s, μ, obtained on the basis of the Papkovich—Neuber representation /3/

$$\begin{aligned} u &= \beta\left(\lambda + \frac{\varkappa}{1+s^2}\right) r\omega_1(s) + \frac{\varkappa r D_1(z, r, s)}{(1+s^2) D_2(z, r, s)} \\ w &= \beta\left(\lambda + \frac{\varkappa}{1+s^2}\right) z\omega_3(s) + \frac{\varkappa z D_1(z, r, s)}{s^2 D_2(z, r, s)} \\ \beta &= 2(1 - 2\nu) \end{aligned} \tag{11}$$

The relationships (G is a scale factor)

$$\frac{b^2}{a^2} = \frac{s_0^2}{1 + s_0^2}, \quad b \leqslant a, \quad r = G\sqrt{(1-\mu^2)(1+s^2)}, \quad z = Gs\mu \tag{12}$$

are valid for the axes a, b of the ellipsoid.

The functions $\omega_1(s)$, $\omega_3(s)$ have the form

$$\begin{aligned} 2\omega_1(s) &= \operatorname{arcctg} s - \frac{s}{1+s^2} \\ \omega_3(s) &= \frac{1}{s} - \operatorname{arcctg} s \end{aligned} \tag{13}$$

The specific form of the functions D_1, D_2 is not needed.

The coefficients $\lambda, \varkappa$ are determined from the system of equations

$$\begin{aligned} \frac{q_1}{4G} &= \frac{\beta(\omega_1 + \omega_3)}{2} - \varkappa\left[\frac{\omega_1}{1+s_0^2} + \nu\frac{\omega_3}{s_0^2} + \frac{(2\nu - 4)s_0^2 - 1}{2s_0^3(1+s_0^2)^2}\right] \\ \frac{q_3}{4G} &= \beta\omega_1 + \varkappa\left[\omega_1\left(\frac{1-\nu}{s_0^2} - \frac{\nu}{s_0^2+1}\right) + \frac{1}{2s_0(1+s_0^2)^2}\right] \end{aligned} \tag{14}$$

The values of ω_1, ω_3 are taken for $s = s_0$.

It is seen that the expressions (11) satisfy condition (8) for $s = s_0$ if $\varkappa = 0$. It follows from (12)—(14) that this is possible for such a ratio of the axes when the quantity s_0 is a root of the equation

$$\frac{1}{s_0} + \left(\frac{q_1}{q_3} - \frac{1}{2}\right)\frac{s_0}{1+s_0^2} - \left(\frac{q_1}{q_3} + \frac{1}{2}\right)\operatorname{arcctg} s_0 = 0 \tag{15}$$

The dependence (12), (15) of the quantity b/a on the ratio q_3/q_1 is shown in Fig.2 (curve *1*). It is seen that equal-strength surfaces exist in the whole range $q_1 \geqslant q_3 \geqslant 0$, not excluding the value $q_3 = 0$. In this case a circular slot is optimal and of equal-strength, as can be confirmed by solving the direct problem for it /3/. For $q_1 = q_3$ the surface Λ becomes a sphere, and the relationship (9) goes over into the known result $\sigma_0 = \sigma_t = {}^3/_2 q_1$ /7/.

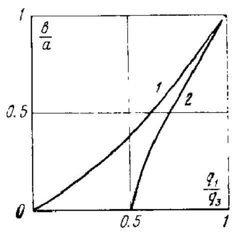

Fig.2

Let us recall that the ratio between the axes of the optimal ellipse in the plane problem is simply equal to the ratio of the loads $b/a = q_3/q_1$ /5/.

When the load along the axis of rotation is greater in magnitude than the load in the latitudinal plane, a prolate spheroid is optimal. In this case the functions $\omega_1(s)$, $\omega_3(s)$ have the form

$$\omega_1(s) = \frac{1}{2}\ln\frac{s+1}{s-1} - \frac{1}{s}$$
$$2\omega_3(s) = \frac{s}{s^2-1} - \frac{1}{2}\ln\frac{s+1}{s-1}$$

Transposing the load notation, we obtain the following dependence from the condition $\varkappa = 0$

$$\frac{1}{s_0} + \left(\frac{q_3}{q_1} - \frac{1}{2}\right)\frac{s_0}{s_0^2-1} - \frac{1}{2}\left(\frac{q_3}{q_1} + \frac{1}{2}\right)\ln\frac{s_0+1}{s_0-1} = 0$$
$$\frac{b}{a} \leqslant 1, \quad \frac{q_3}{q_1} \leqslant 1, \quad \frac{b^2}{a^2} = \frac{s_0^2-1}{s_0^2}$$

represented by curve *2* in Fig.2. The solution of the optimization problem exists only under the additional constraint

$$2q_3/q_1 \geqslant 1$$

If $w(r,z)$ is identified with the velocity potential $\varphi(r,z)$, and $ru(r,z)$ with the Stokes function $\Psi(r,z)$ taken with the opposite sign /8/, then the problem under consideration admits the following hydrodynamic analogy: find the shape of the surface Λ of an axisymmetric system of solids around which a steady ideal fluid flow streams along the z axis under the condition that its velocity is given at infinity

$$V_\infty = {}^1/_2(q_2 + p)$$

and on Λ

$$V = {}^1/_2(2q_1 + q_2 + p)\,dz/ds$$

For $n > 1$ this permits utilization of numerical methods of hydrodynamics to find Λ. A similar analogy was noted for the plane problem in /5/.

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